Appendix to “Inflating Away the Public Debt? An Empirical Assessment”

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\section{Proof of proposition 1}

While the bulk of the proof is already in section 2, here we fill in some missing steps. First, adding and subtracting $Q_t^1 W_{t+1}$ on the right-hand side of equation (2), and using equation (1) to replace out $W_{t+1}$, a few steps of algebra deliver the law of motion for the market value of government debt:

$$W_t = Q_t^1 W_{t+1} + s_t + x_{t+1} \quad (A1)$$

where the revaluation term $x_{t+1}$ is equal to:

$$\sum_{j=0}^{\infty} (Q_{t}^{j+1} - Q_{t}^{j} Q_{t+1}^{j}) K_{t+1}^j + \sum_{j=0}^{\infty} (H_{t}^{j+1} - H_{t}^{j} H_{t+1}^j) \frac{B_{t+1}^j}{P_t} + \sum_{j=0}^{\infty} \frac{H_{t+1}^{j+1} B_{t+1}^j}{P_t+1} \left( \frac{P_{t+1} H_{t}^j}{P_t} - Q_{t}^j \right). \quad (A2)$$

Iterating this equation forward, from date 0 to date $t + 1$, delivers equation (3) in the text.

Dividing both sides of the law of motion for $W_t$ by $Q_t^1$, multiplying by $m_{t,t+1}$ and taking
expectations gives:

\[ W_t = \mathbb{E}_t(m_{t,t+1}W_{t+1}) + s_t + \mathbb{E}_t \left( \frac{m_{t,t+1}x_{t+1}}{Q^1_t} \right). \] (A3)

For now, assume that the last term on the right-hand side is zero. We will show it shortly. Multiply both sides of (A3) by \( m_{0,t} \) and take expectations as of date 0, so that using the law of iterated expectations you get the recursion:

\[ \mathbb{E}_0(m_{0,t}W_t) = \mathbb{E}_0(m_{0,t+1}W_{t+1}) + \mathbb{E}_0(m_{0,t}s_t). \] (A4)

Iterate this forward from date 0 to date \( T \), and take the limit as \( T \) goes to infinity. With the no-Ponzi scheme condition \( \lim_{T \to \infty} \mathbb{E}_0(m_{0,T}W_T) = 0 \), you get the result in expression (5):

\[ W_0 = \mathbb{E}_0 \left[ \sum_{t=0}^{\infty} m_{0,t}s_t \right]. \] (A5)

Finally, for the first equality in expression (5), replace out the bond prices from equation (1) using equation (4).

The missing step was to show that \( \mathbb{E}_t(m_{t,t+1}x_{t+1}/Q^1_t) = 0 \) for all \( t \). Consider the first element of \( x_{t+1} \) and take the following steps:

\[ \mathbb{E}_t \left[ \sum_{j=0}^{\infty} (Q^j_{t+1} - Q^j_{t+1}) K^j_{t+1} \right] = \]

\[ \sum_{j=0}^{\infty} K^j_{t+1} \left( Q^j_{t+1} - \mathbb{E}_t \left( m_{t,t+1} x_{t+1}/Q^1_t \right) \right) = \]

\[ \sum_{j=0}^{\infty} K^j_{t+1} \left( Q^j_{t+1} - \mathbb{E}_t \left( m_{t,t+1} x_{t+1} + j \right) \right) = \]

\[ \sum_{j=0}^{\infty} K^j_{t+1} \left( Q^j_{t+1} - \mathbb{E}_t \left( m_{t,t+1} + j \right) \right) = 0. \] (A6)
The first equality comes from replacing the expectation of the sum by the sum of the expectations and from taking the prices known at date $t$ out of the expectation; the second from using the result in equation (4) twice; the third from using the law of iterated expectations; and the fourth from using equation (4). Identical steps show that the other terms in $\mathbb{E}_t(m_{t,t+1}x_{t+1}/Q_t^1)$ are also equal to zero.

The final part of the proof to clarify is expression (6). First note that in principle, $m_{0,t}$ can depend on many random variables. However, all we need to evaluate is $\mathbb{E}(m_{0,t}/\pi_{0,t})$. Therefore, only the dependence of $m_{0,t}$ on $\pi_{0,t}$ will lead to a non-zero term once the expectation of the product of the discount factor and inverse inflation is evaluated. Therefore, we can write $\mathbb{E}(m(\pi_{0,t})/\pi_{0,t})$.

Second, let $\hat{f}(\cdot)$ be the probability density function for inflation. Then, we can re-write the expression as: $\int \left( \hat{f}(\pi_{0,t})m(\pi_{0,t})/\pi_{0,t} \right) d\pi_{0,t}$. Define the inverse of the risk-free rate, from the perspective of date 0, as $R_t^{-1} = \int \hat{f}(\pi_{0,t})m(\pi_{0,t})d\pi_{0,t}$. From the definition of the stochastic discount factor, this is the price of a bond that would pay one dollar for sure in $t$ periods, regardless of the realization of inflation. Then, define the risk-adjusted density for inflation as: $f(\pi_{0,t}) = \hat{f}(\pi_{0,t})m(\pi_{0,t})R_t$. Using these results it follows that:

$$
\mathbb{E}(m(\pi_{0,t})/\pi_{0,t}) = \int \left( \frac{\hat{f}(\pi_{0,t})m(\pi_{0,t})}{\pi_{0,t}} \right) d\pi_{0,t} = R_t^{-1} \int \left( \frac{f(\pi_{0,t})}{\pi_{0,t}} \right) d\pi_{0,t}. \quad (\Lambda 7)
$$

This confirms expression (6).

**B  Debt holdings**

We construct monthly maturity structures, that is $B^t_0$, for four groups of investors: (1) private total, which is publicly held debt, excluding the Federal Reserve and state and local government holdings, (2) foreign, (3) domestic, which is private minus foreign, and (4) the
central bank. The data sources are the CRSP U.S. Treasury database, the Treasury Bulletin, the “Foreign Portfolio Holdings of U.S. Securities” report available from the U.S. Treasury, and the System Open Market Account (SOMA) holdings available from the FRBNY. The numbers for face value of total outstanding debt for different categories in section 3.1 come from the Monthly Statement of Public Debt, the Treasury Bulletin (OFS-2, FD-3), and the SOMA. Data are all for end of December 2015 for the United States.

We construct holdings of notes and bonds as follows. For (1) the data is available from CRSP, and we subtract out state and local government holdings assuming they have the same maturity structure. Detailed information for (2) is available for June 2015 and we assume proportionate growth of foreign holdings to construct December 2015 numbers. (3) is the difference of (1) and (2). For (4) we use the security level data available from the FRBNY. We assume that all coupon and principal payments mature in the middle of each month.

CRSP does not have data on Treasury bills. We use the issues of the Treasury bulletin to obtain information on bills and follow the same steps as we did above for notes and bonds. In particular, we construct holding of T-Bills as follows. For (1) we subtract Fed holdings from Treasury Bulletin Table FD-2 T-Bill holdings and then subtract out the proportionate amount of state and local government holdings; for (2) and (3) we do the same as for notes and bonds; for (4) we use the SOMA holdings.

Aside from calculating debt holdings for more categories of investors, our method for constructing the maturity structure of (1) has the following differences relative to Hall and Sargent (2011). First, we construct a monthly term structure and assume that promised payments in a month are paid in the middle of the month (instead of using an annual frequency). Second, we exclude state and local government holdings. Third, we base the T-Bill holdings on Table FD-2 of the Treasury bulletin and Fed holdings (rather than Table FD-5 of the Treasury bulletin).
Real interest rates are constructed according to equation (8) in the paper and the spot curve (nominal zero coupon yield curve) provided by Gürkaynak, Sack and Wright (2007). The harmonic means of inflation are calculated using simulated 30-year distributions based on our fitted restricted model. As a check, we also separately calculate the market value of the debt using the real term structure from Gürkaynak, Sack and Wright (2010) for real interest rates. These only extend up to 20 years. To construct real spot rates for longer maturities we assume that forward rates for years 21 to 30 are equal to the average forward rate for years 18 to 20. Our results are robust to this alternative.

C Estimating the marginal distributions for inflation

We estimate inflation distributions from data on zero coupon and year-on-year inflation caps and floors, which we collect from Bloomberg, as do Fleckenstein, Longstaff and Lustig (2017). Kitsul and Wright (2013) use data provided by an interdealer organization, whereas we use the raw reported numbers.

C.1 Zero-coupon inflation options

The basic methodology for construction of the distributions from zero coupon inflation options is fairly standard and similar to Kitsul and Wright (2013) and Fleckenstein, Longstaff and Lustig (2017). Still, we take several steps in cleaning the data that this section of the appendix clarifies.

The zero coupon inflation call options are traded with strike prices between 1% and 6%, in 0.5% increments, and expiration dates ranging from 1 to 10 years as well as 12 and 15 years, although the data for the 2 and 9 year maturities are of generally lower quality. The zero coupon put options are available for strike prices between -2% and 3% and identical maturities as the caps. For the overlapping range of strike prices we use both option prices
to reduce measurement error.

A zero coupon inflation cap is the most traded contract among inflation derivatives. It pays, at expiry, the maximum between zero and the difference between the cumulate inflation during the period and the strike price so its payoff at maturity is $\max[0, (1 + \pi_{0,t}) - (1 + k)t]$. Adapting the argument in Breeden and Litzenberger (1978) we can non-parametrically construct risk neutral density functions using these option prices.

In particular, for our options, the asset pricing formula is:

$$a_0 = R_t^{-1} \mathbb{E}_f(\max\{(P_t - k)/P_t, 0\}).$$

where $a_0$ is the price of the bond, $R_t$ is the real safe return, $f(.)$ is the real risk-neutral measure, $k$ is the strike, while $P_t$ is the price index, with $P_0 = 1$ so $\pi_{0,t} = P_t$. Taking the derivative of the pricing equation with respect to $k$ gives:

$$R_t \frac{\partial a_0}{\partial k} = - \int_k^\infty \frac{1}{P_t} f(P_t) dP_t.$$

Then take a second derivative

$$R_t \frac{\partial^2 a_0}{\partial k^2} = \frac{f(k)}{k}.$$

Therefore, to get the real risk neutral measure of inflation, one needs to calculate:

$$f(\pi_{0,t}) = \pi_{0,t} R_t \frac{\partial^2 a_0}{\partial \pi_{0,t}^2}$$

We can extract the risk-neutral density by observing how the price of the option varies with changes in the strike price.

Breeden and Litzenberger (1978) suggest using a butterfly trading strategy to construct Arrow-Debreu securities, claims that pay one unit of currency if at some specific time in
the future the underlying asset price is equal to a specific value and zero otherwise. While this method provides a good first approximation to risk neutral probabilities, it does not adjust for irregular options prices, due to, for example, non-synchronous trading (Bahra, 1997). Therefore, while we check that all such prices are positive, we must smooth the data otherwise the results are very inaccurate.

To overcome the drawbacks of the unadjusted butterfly strategy, we do the following. First, we drop data that represent simple arbitrage opportunities (discussed in the text). We next calculate Black-Scholes implied volatilities, and, following Shimko (1993) and Campa, Chang and Reider (1998), for each set of options at any expiry date, we fit a four factor stochastic volatility model, the SABR model, which was developed by Hagan et al. (2002). This calibration minimizes the norm of the difference between the observed data and the candidate SABR function, resulting in a vector of optimal parameters for the SABR model which is used to create a volatility curve. We constrain the estimated implied volatility function to ensure that the smoothing does not re-introduce arbitrage opportunities. This method reduces the weight of irregular data, while preserving its overall form. We convert back to option prices and construct risk neutral distributions.

To be clear, this method does not assume that we can use the Black-Scholes formula for pricing. Instead, it is simply used as a nonlinear transformation on which smoothing is performed.

C.2 Year-on-year inflation options

The construction of distributions from year-on-year options requires a bootstrapping method where cap and floor contracts, which are portfolios of annual caplets and floorlets, are unbundled to recover prices of the underlying options. First, we use a bootstrapping procedure to extract the caplet and the floorlet prices from the cap and floor prices respectively. Second, when calculating the option’s implied volatility we use the Rubinstein (1991) transformation
which enables us to price the option as a plain vanilla option with a time to maturity equal to the option tenor between inflation reset times, discounting back using the real interest rate, calculated assuming that put-call parity holds for these options (Birru and Figlewski, 2012).

For each horizon, for the smallest (largest) bin we report the risk-adjusted probability of inflation lying in or below (above) that bin. Because there was considerable mass in the last bin (5.5% to 6% inflation) for year-on-year inflation, we use our smoothed estimates to project four additional bins.

C.3 Comparing option prices at different dates

We inspect option prices at the end of December 2015 and start of January of 2016 (our data of interest) and selected out of the last five trading days of the year and the first five trading days of the new year, the day that yields the maximum number of option prices that do not violate the threshold criteria, described above and in the text. This was January 5, which is the benchmark for our calculations. We looked at distributions for the 5 days before and after. All of them looked almost identical to the ones we used.

More interesting, we also constructed inflation distributions one month before and after. Figure 1 shows that while there are some changes in the 1 and 2 year maturities, they are small. All the results in the paper are therefore robust to using instead the distributions for one month before or one month after.

D Estimating the joint distributions for inflation

The marginal distributions for inflation are enough to evaluate the formula in proposition 1. Yet, to calculate inflation paths or counterfactuals, we need joint distributions.
Figure 1: Marginal distribution for risk-adjusted cumulative inflation at three dates
D.1 Proof of proposition 2

Using $F(.)$ to denote the cumulative density function, we have data for one-year inflation $F(ln \pi_{t+j-1,t+j})$ for $j = 1...J$ where $J = 10$ years, and for cumulative inflation $F(ln \pi_{t,t+j})$. The data comes in $N$ bins expressed as ranges for inflation.

Sklar’s theorem states that there exists a function $c : [0, 1]^J \rightarrow [0, 1]$ such that:

$$F(ln \pi_{t,t+1}, ..., ln \pi_{t+J-1,t+J}) = c(F(ln \pi_{t,t+1}), F(ln \pi_{t+1,t+2}), ..., F(ln \pi_{t+J-1,t+J})). \quad (A8)$$

In turn, it follows from the link between marginal and joint distributions and the definition of cumulative inflation that:

$$F(ln \pi_{t,t+j}) = \int_{\Pi} F(ln \pi_{t,t+1}, ln \pi_{t+1,t+2}, ..., ln \pi_{t+J-1,t+J}) dln \pi_{t,t+1}...dln \pi_{t+J-1,t+J} \quad (A9)$$

where the set $\Pi$ is defined as: $\{ln \pi_{t,t+j} : ln \pi_{t,t+n} \sum_{i=1}^{j} ln \pi_{t+i-1,t+i}\}$.

Combining these two results delivers the proposition in terms of cumulative distributions. We maintain the assumption throughout that all distributions are continuous. Therefore, Sklar’s theorem also applies to the marginal density functions, with $C(.)$ replacing $c(.)$. For numerical purposes, it was better to work with densities rather than with cumulative distribution functions.

A final point to note is that there are $N$ bins and so $N$ equalities in the distribution, of which one is redundant since probabilities must add up to one. There are $J$ maturities, but for maturity one the equality is trivial. Therefore, in total there are $(N - 1)(J - 1)$ conditions.
D.2 Parametric copula and method of moments

We use the parametric normal copula, whose formula is:

\[
\hat{C}(f(\ln \pi_{t,t+1}), \ldots, f(\ln \pi_{t+J-1,t+J})) = \\
\left( \frac{1}{\det \hat{R}} \right) \exp \left( -\frac{1}{2} \begin{pmatrix} 
\Phi^{-1}(f(\ln \pi_{t,t+1})) \\
\vdots \\
\Phi^{-1}(f(\ln \pi_{t+J-1,t+J})) 
\end{pmatrix} \left( \rho^{-1} - \mathbf{I}_J \right) \begin{pmatrix} 
\Phi^{-1}(f(\ln \pi_{t,t+1})) \\
\vdots \\
\Phi^{-1}(f(\ln \pi_{t+J-1,t+J})) 
\end{pmatrix} \right) 
\]  

where \( \Phi(.)^{-1} \) is the inverse of the standard normal cdf, and \( \rho \) is a correlation matrix of dimension \( J \).

The matrix \( \rho \) would only exactly equal the correlation matrix of the variables in the joint distribution if the marginal distributions happened to be normal. Yet, by drawing from the joint distribution using the formula above, and calculating correlation coefficients across many draws, we found that the difference between the actual correlation matrix and \( \rho \) was almost always less than 0.01.

To find the unrestricted estimates, we minimize over the \( J(J-1)/2 = 45 \) independent components of the correlation matrix \( \rho \) that lie between \(-1\) and \(1\). The objective function is the equally weighted average of the \( (N-1)(J-1) = 20 \times 9 = 180 \) squared deviations from the moments in proposition 2. This is a difficult global minimization over a large parameter space, which we handle through a combination of global and local minimization algorithms and many repeated searches from Sobol-sampled starting points.

D.3 Inflation after 10 years

In drawing paths for inflation, for the first 10 years we use the joint distribution given by the multivariate copula. After that, we assume that inflation is a 9th order Markov process. Therefore, the distribution for inflation in year 11, conditional on inflation in years 2 to 10
is the same as the distribution for inflation in year 10, conditional on years 1 to 9. Since we have the joint distribution for inflation from year 1 to 10, it is easy to derive the conditional distribution for inflation in year 10, conditional on years 1 to 9. Thus, we have the conditional distribution for year 11, conditional on the draws so far. The same applies to year 12, and so on, all the way to 30. Note that, since we assumed that the joint distribution of maturities one to ten follows a Gaussian copula, then this procedure implicitly assumes that the joint distribution in maturities one to thirty is likewise a Gaussian copula. The key restriction is that the correlation matrix of $30 \times 29/2 = 435$ elements for the 1-30 copula only have 45 independent separate elements that we estimated for the 1-10 distribution.

### D.4 Restricted distribution

We can represent inflation between two successive dates (or maturities) as:

$$\ln \pi_{t,t+1} = \mathbb{E}_t(\ln \pi_{t,t+1}) + p_{t+1} + s_{t+1},$$

where $p_{t+1}$ is a non-stationary part, and $s_{t+1}$ a stationary one, with the two being independent and zero mean. From Wold’s theorem, the stationary process is fully characterized by its covariance function $v_j = \mathbb{E}_t(s_{t+n}s_{t+n+j})$ for any arbitrary $j$. The definition of stationary is that this is true for any positive $n$.

The restriction that we impose is that the non-stationary process is a random walk, so that $\mathbb{E}_t(p_{t+n}p_{t+n+j}) = \sigma n$. This constrains the way in which non-stationarity affects the correlation matrix over time. In particular, the correlation between inflation at date $t + n$ and date $t + j$, or the $(n, j)$ element of the matrix $\rho$ is given by the expression:

$$\rho_{n,n+j} = \frac{v_j + n\sigma}{\sqrt{(v_0 + n\sigma)(v_0 + (n + j)\sigma)}}$$

(A12)
While the unrestricted $\rho$ matrix has 45 parameters, with the random-walk constraint, there are only 10 parameters. Nine are the correlations of the stationary part \( \{ v_1/v_0, ..., v_9/v_0 \} \), and one more is the ratio of the relative variances of the permanent and transitory components $\sigma/v_0$. We minimize the same objective function but over this smaller parameter space. Note that in this case, the $\rho$ matrix has the easily identified form, $\rho_{n,n+j} = v_j/v_0$, which is the same whatever is $n$.

## D.5 Goodness of fit and restricted versus unrestricted distribution

Table 1 shows the estimated $\rho$ for the unrestricted model. Because the estimates vary so much, they are a little hard to interpret. Therefore, in Table 2 we calculate the average correlation coefficients between short term (1 to 3 years), medium term (4 to 7 years) and long term maturities (8 to 10 years). The two key features that are in common with the restricted estimates are the ones emphasized in the text: the autocorrelation coefficients are not very high, and they do not fall with maturity. But the estimates are quite noisy in that, across contiguous maturities, they jump up and down a bit.

We assess goodness of fit in multiple ways. First, figure 2 shows the data for the risk-neutral density of cumulative inflation at maturities 2 to 10 against the predicted densities, according to the restricted and unrestricted models. Our method of moments, following proposition 2, consisted of picking parameters $\rho$ to minimize the difference between the data in these 9 plots, and the models. Visually the fit is quite good, and the restriction seems to have almost no effect on the ability of the copula model to fit the data.

Second, we compare the models’ prediction for inflation at horizons 12 and 15 with the data for those maturities. Note that this tests not only the normal copula, but also our 9th order Markov assumption to simulate beyond 10 years. Figure 3 shows the model-implied and data distributions. The restricted model fits better.

Third, we compare the model’s predicted standard deviations of risk-adjusted inflation,
Table 1: Estimated correlation coefficients of year-on-year inflation in the joint distribution

<table>
<thead>
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<th>Maturity</th>
<th>1</th>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
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Notes: Estimated correlation coefficients for year-on-year inflation between date 2015+j and 2015+I, in column j, row l.

Table 2: Average correlation coefficients of year-on-year inflation in the joint distribution

<table>
<thead>
<tr>
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<th>Short term (1-3 years)</th>
<th>Medium term (4-7 years)</th>
<th>Long term (8-10 years)</th>
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<td>Medium term (4-7 years)</td>
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<td>Long term (8-10 years)</td>
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<td>0.72</td>
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Figure 2: Marginal distribution for risk-adjusted cumulative inflation: data and models

Figure 3: Distribution for $\ln \pi_{t,t+11}$ and $\ln \pi_{t,t+15}$, model and data
with those in the marginal distributions. Figure 4 shows the two models. The restricted model again seems to do a slightly better job when applied to the horizon 12 and 15 distributions, although again it is clear that, in spite of the slightly different estimates of $\rho$, the two models have quite similar fits.

Fourth, we repeated our calculations for one month after and one month before, to assess by how much the estimates of rho change across the two time intervals. Table 3 shows the resulting estimates for the restricted model only, to save space. While the estimates change a little, the overall pattern that we describe in the text is very similar. Again, consequently, all our results are robust to using the one month before or after distributions instead.

**E Estimating debt debasement**

We draw 500,000 samples for 40 years of inflation using our joint distribution. We convert these into 480 month histories by assuming continuous compounding and a constant inflation
Table 3: Estimates of $\rho$ one month before and after Horizon Baseline (Dec 2015) Nov 2015 Jan 2016

<table>
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<td>0.17</td>
<td>0.18</td>
</tr>
<tr>
<td>7</td>
<td>0.25</td>
<td>0.22</td>
<td>0.26</td>
</tr>
<tr>
<td>8</td>
<td>0.35</td>
<td>0.35</td>
<td>0.32</td>
</tr>
<tr>
<td>9</td>
<td>0.79</td>
<td>0.76</td>
<td>0.79</td>
</tr>
</tbody>
</table>

Notes: Estimated restricted correlation coefficients for year-on-year inflation

rate within each year. We calculate the real value of the nominal payments for each of these draws, and order them, calculating their percentiles for table 2. We repeat the calculations using the $B_0^t$ for each group of investors instead.

For the counterfactuals, we use alternative distributions for inflation from which to take draws, recalculate the value of the debt using the formula in proposition 1, and subtract it from the original number. The alternative distributions are:

1. Permanently higher: Shift all the year-on-year distributions by the difference between the 90th and the 50th percentile at each maturity.

2. Right tail only: We draw from the baseline distribution but discard all histories in which average inflation is below its 90th percentile in any one of the years in the simulation.

3. Higher and more variable: For each maturity we multiply baseline inflation levels by a scaling factor so that the new mean is equal to the mean in case 1. This results in
more variable inflation.

4. Higher for sure: This is the same as case 1 but we now assign all the weight to the mean at each maturity.

5. Partially expected: We set inflation equal to 3% in year 1. For the following histories we use marginal distributions conditional on the year 1 realization. After 10 years we assume that year-on-year inflation is 9th order Markov.

6. Temporary increase: We shift the year-on-year distributions in the same way as case 1 but we now shift them to the 90th percentile in year 1, 80th in year 2, 70th in year 3, and 60th in year 4. There is no change in the year-on-year distributions for maturities equal to and above five years. Note that this is different from the unexpected shock in the previous case since distributions in years 2 to 4 are shifted directly instead of changing only due to the new distribution in year 1.

7. Gradual increase: The one year distribution is unchanged, the 2 year median shifts to the previous 60th percentile, 70th for year 3, 80th in year 4, and 90th for 5 years and above.

F Proof of proposition 3

Financial repression consists of paying nominal bonds that are due at date \( t \) with new \( N \)-period special debt that sells for the price \( \bar{H}_t^N \):

\[
B_t^0 = \bar{H}_t^N \bar{B}_{t+1}^{N-1}.
\]  

(A13)
The value of outstanding debt at date $t$ now is:

$$W_t = \sum_{j=0}^{\infty} \frac{H_j^t B_j^t}{P_t} + \sum_{j=0}^{\infty} Q_j^t \tilde{K}_j^t + \frac{\tilde{B}_{t-N+1}^N}{P_t},$$

(A14)

while the government budget constraint now is:

$$W_t = s_t + \sum_{j=0}^{\infty} \frac{H_{j+1}^t B_{j+1}^t}{P_t} + \sum_{j=0}^{\infty} Q_{j+1}^t \tilde{K}_{j+1}^t + \frac{\tilde{H}_t^N \tilde{B}_{t+1}^{N-1}}{P_t}.$$

(A15)

The text covered the special case where $N = 1$, there is no real debt, and all nominal debt had one period maturity. This appendix proves the general case.

Combining these two equations just as we did in the proof of proposition 1, we end up with a law of motion for debt:

$$W_t = Q_t W_{t+1} + s_t + x_{t+1} + \frac{\tilde{H}_t^N \tilde{B}_{t+1}^{N-1}}{P_t} - Q_t \tilde{B}_{t-N+2}^N.$$

(A16)

By precisely the same steps as in the proof of proposition 1, it then follows that:

$$W_0 = \mathbb{E} \left[ \sum_{t=0}^{\infty} m_{0,t} \left( \frac{B_0^t}{P_t} \right) \right] + \mathbb{E} \left[ \sum_{t=0}^{\infty} m_{0,t} \tilde{K}_0^t \right] = \mathbb{E} \left[ \sum_{t=0}^{\infty} m_{0,t} \left( s_t + \frac{\tilde{H}_t^N \tilde{B}_{t+1}^{N-1} - Q_t \tilde{B}_{t-N+2}^N}{P_t} \right) \right].$$

(A17)

Now, replacing the new debt with old debt using equation (A13), we get that the nominal debt burden then becomes:

$$\mathbb{E} \left[ \sum_{t=0}^{\infty} m_{0,t} \left( \frac{B_0^t}{P_t} \right) \right] - \mathbb{E} \left[ \sum_{t=0}^{\infty} m_{0,t} \left( \frac{B_0^t}{P_t} \right) \right] + \mathbb{E} \left[ \sum_{t=0}^{\infty} m_{0,t} \frac{Q_t \tilde{B}_{t-N+1}^N}{P_t} \right].$$

(A18)

Canceling terms and relabeling the limits of the sums (since financial repression started at
date 0) we get the nominal debt burden:

\[
\mathbb{E}\left[ \sum_{t=0}^{\infty} m_{0,t+N-1} \frac{Q_{t+N-1}^1 B_0^t}{\hat{H}_t^N P_{t+N}} \right].
\]  

(A19)

Finally, recall that \( Q_{t+N-1}^1 = \mathbb{E}_{t+N-1}(m_{t+N-1,t+N}) \). Using the law of iterated expectations we end up with:

\[
\mathbb{E}\left[ \sum_{t=0}^{\infty} m_{0,t+N} \left( \frac{1}{\pi_{t,t+N} \hat{H}_t^N} \right) \left( \frac{B_0^t}{P_t} \right) \right].
\]  

(A20)

To get the equality in the proposition, simply use the upper bound \( H_t^N = 1 \). To get the equality written in terms of the price of nominal bonds, simply use the law of iterated expectations and the arbitrage condition: \( H_t^N = \mathbb{E}(m_{t,t+N}/\pi_{t+N}) \).

References


